Huygens-Fresnel principle. Fresnel diffraction.

Diffraction is an ensemble of various effects when light propagate differently than predicted by geometrical optics and can go into the geometrical shadow.

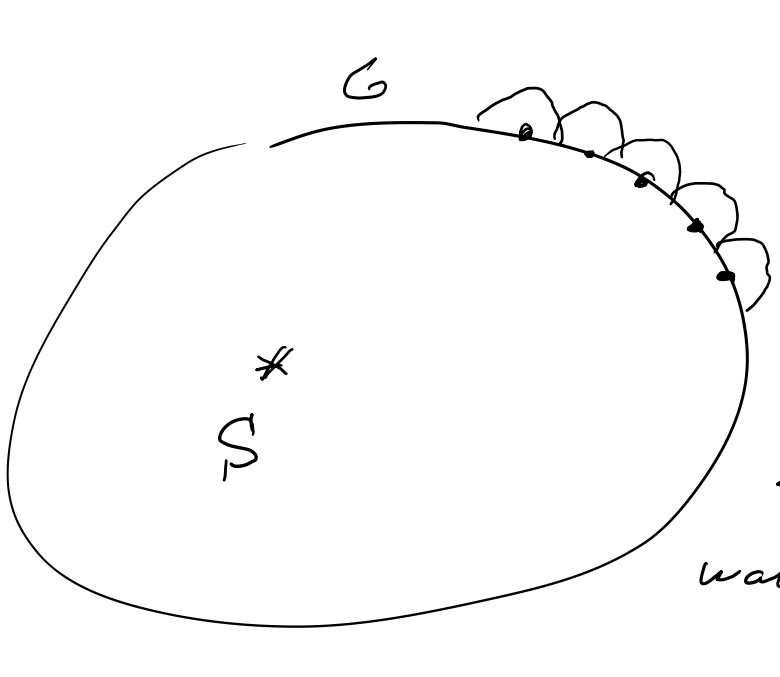
Diffraction problems are formally have to be solved using wave equation with boundary conditions.

$$\nabla^2 E - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = 0 \quad \text{with boundary conditions}$$

However, not for all problems such equation has trivial solution. Usually, such problems are solved using approximation methods, like Huygens - Fresnel principle. 1678 - 1818

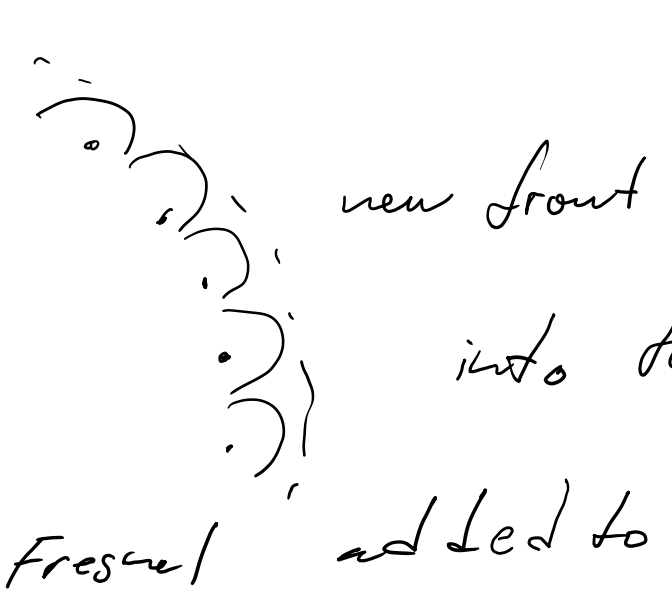
Huygens - Fresnel principle consists of two statements:

① Each point of space that e/m wave reachs becomes secondary source of e/m waves (Huygens)



If one have the source S, then it is possible to take surface G and look at each point as source of secondary spherical waves.

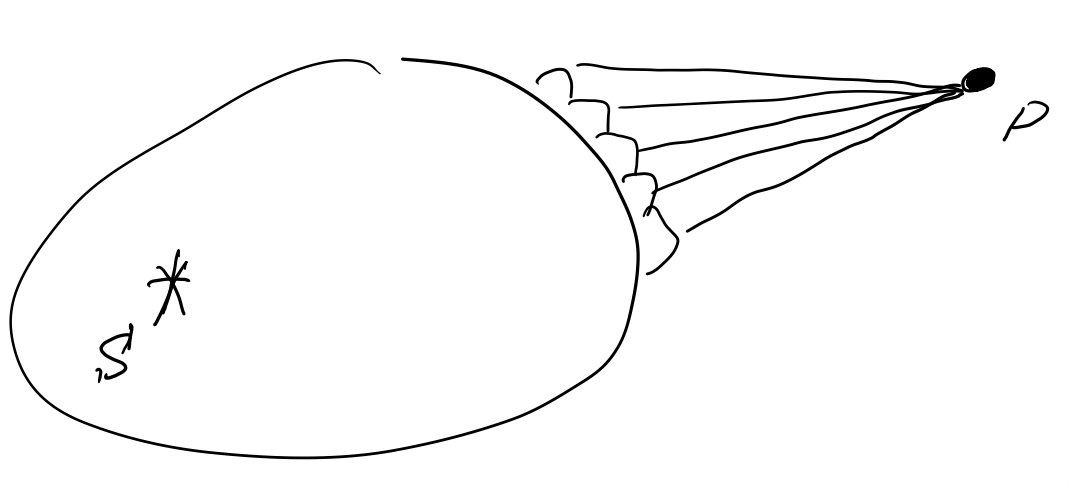
This principle was initially used to explain reflection and refraction laws.



However, it does not explain diffraction, when the light is propagating into the geometrical shadow.

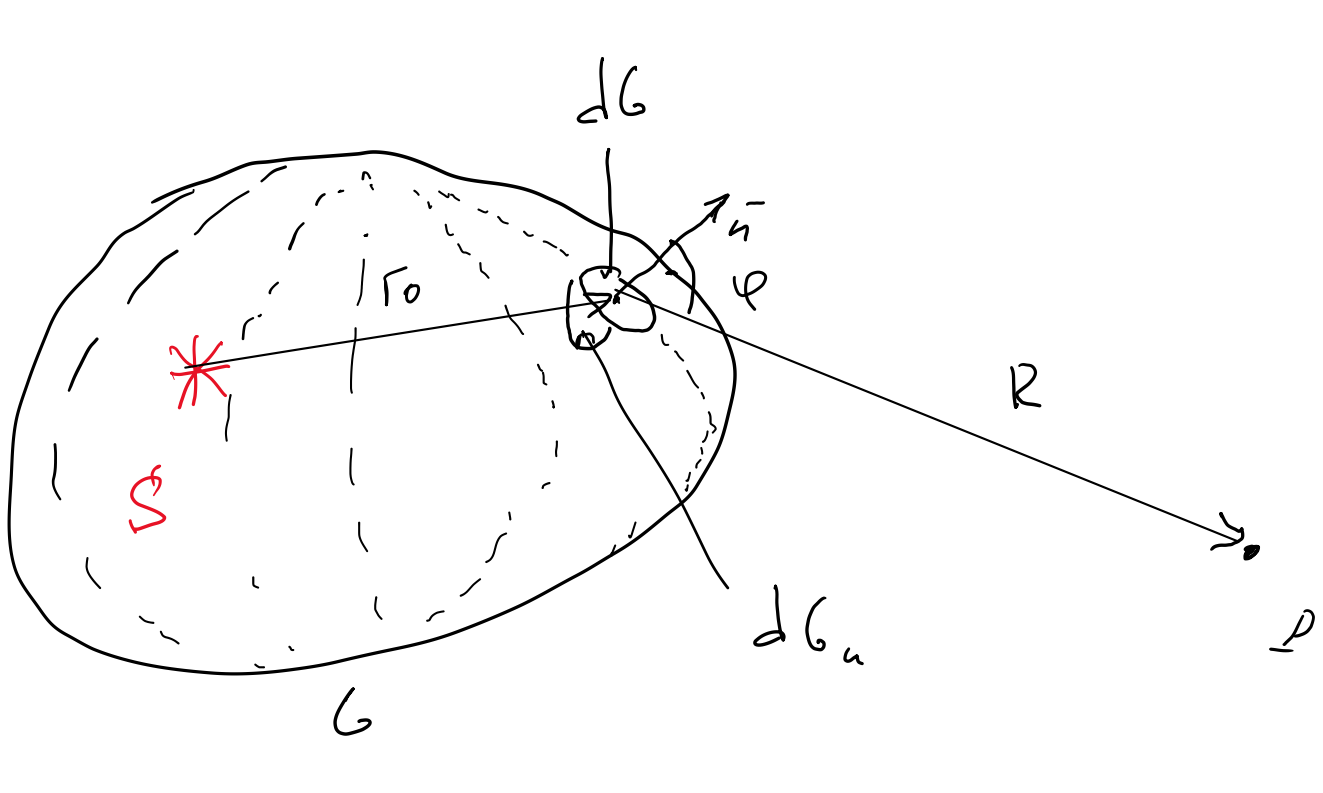
② Fresnel added to this principle statement, that all secondary waves are coherent.

$$E_p = \sum E_i$$



Let's use this principle and solve couple of simple problems.

Mathematical description H-F principle



We have source S and point of observation P. Our goal is to find amplitude of oscillations in point P. For this we will surround S with arbitrary surface G and choose secondary source which surface will be dG.

Amplitude of oscillation of this secondary source:

$$dA = \frac{E_0'}{r_0} e^{ikr_0} dG_n$$

i.e.

1) inverse proportional to r_0
2) phase acquired kr_0
3) E_0' is some constant that is property of S.
4) proportional to dG_n.

Such secondary waves are everywhere on surface G. They radiate waves that arrive at point P and create resulting field:

$$dE_p = K(\varphi)\left(\frac{E_0'}{r_0} e^{ikr_0}\right)\frac{e^{ikR}}{R} dG_n$$

i.e.

1) inverse proportional to R
2) additional phase kR.
3) $K(\varphi)$ is coefficient that reflects anisotropy of radiation of secondary source.

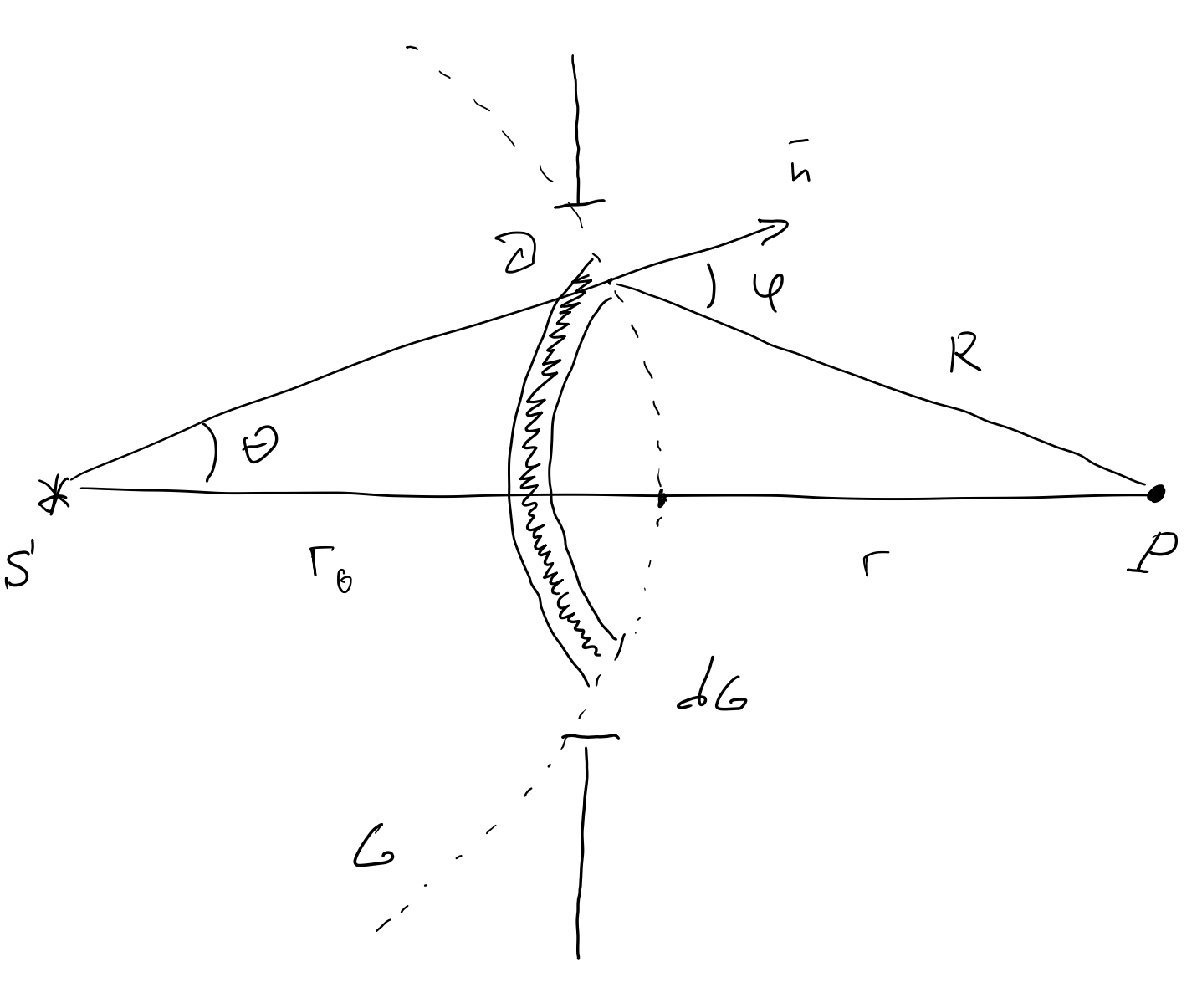
The resulting field:

$$\boxed{E_p = \int_G K(\varphi)\frac{E_0'}{r_0 R} e^{ik(r_0+R)} dG_n}$$

↖ mathematical description of H-F principle

Now, using this principle we will solve few simple problems.

Diffraction on round hole. Fresnel's logic.



We need to find amplitude of oscillation at point P. Line between S and P is going through the center of the hole.

Here, we assume that field at the space, where hole is open is the same as open space field, i.e. the hole boundaries do not influence any field distribution. In real life this is not true as boundaries contain electrons and nuclea, hence can alter field distribution.

G is spherical surface with radius r_0.

$$dG = dG_n = 2\pi r_0 \sin\theta\, r_0\, d\theta$$

From ΔSOP through cos theorem (cos rule) we can determine radius R:

$$R^2 = r_0^2 + (r_0+r)^2 - 2r_0(r_0+r)\cos\theta$$

Let's take derivative:

$$2R\,dR = 2r_0(r_0+r)\sin\theta\, d\theta$$

Here only θ is variable. As we go through G other parameters are constants, i.e. r and r_0.

Hence

$$dG = \frac{2\pi R\, dR\, r_0}{r_0+r}$$

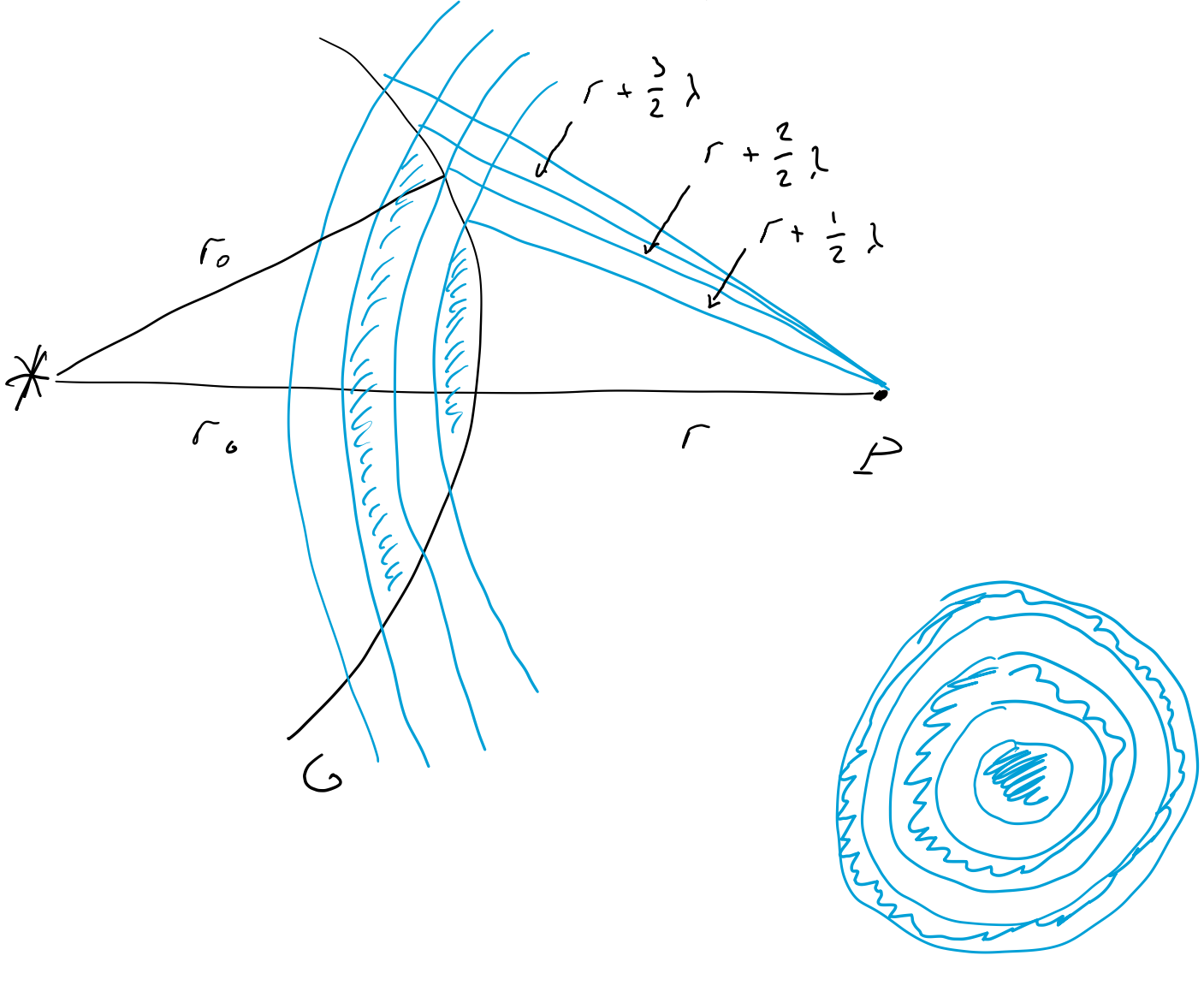
Using mathematical derivation of H-F principle:

$$E_p = \int_G K(\varphi)\frac{E_0'}{r_0 R} e^{ik(r_0+R)} dG_n$$

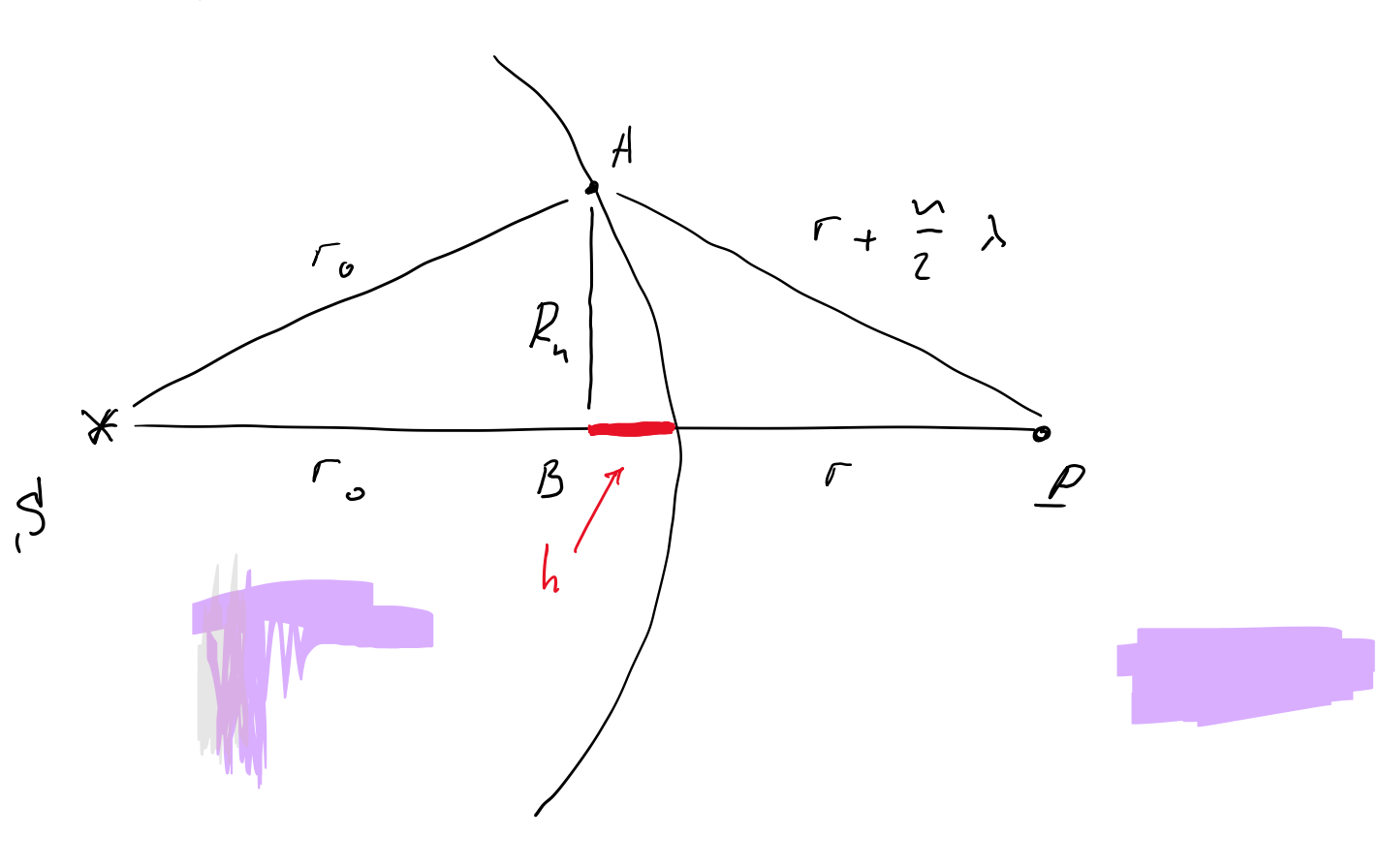
$$\Downarrow$$

$$E_p = \underbrace{2\pi\frac{r_0}{r_0+r}\frac{E_0'}{r_0}e^{ikr_0}}_{\text{const}}\int_r^{R_{max}} K(R)e^{ikR}dR$$

How to take this integral? The problem is that we do not know K(R). Let's introduce Fresnel zones. Fresnel proposed the following.



Let's find the radius of n - zone R_n.



We can derive R_n from triangles SAB and PAB

$$\overset{SAB}{R_n^2 = r_0^2 - (r_0-h)^2} = \overset{PAB}{\left(r + n\frac{\lambda}{2}\right)^2 - (r+h)^2}$$

If we drop $\left(n\frac{\lambda}{2}\right)^2$, then

$$2r_0 h = nr\lambda - 2rh$$

$$\Downarrow$$

$$h = \frac{nr\lambda}{2(r_0+r)}$$

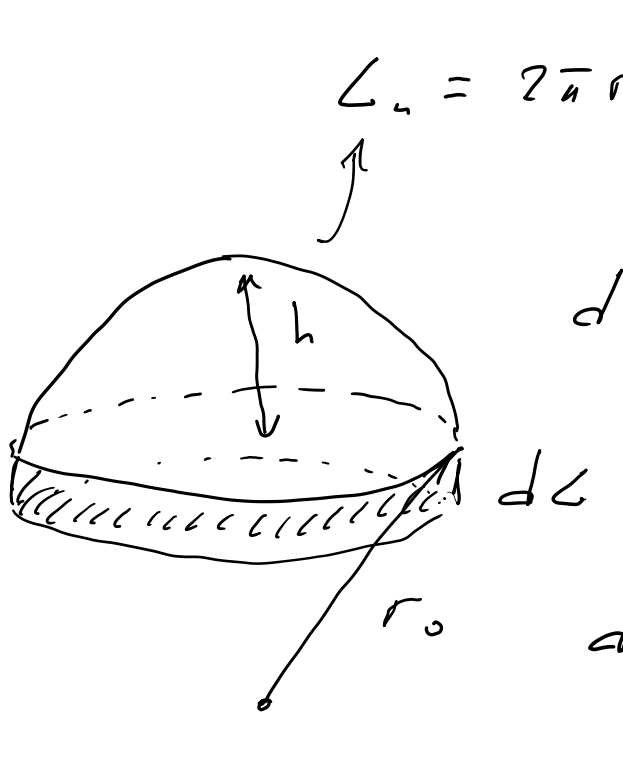
$$R_n = \sqrt{r_0^2-(r_0-h)^2} = \sqrt{2r_0h} = \sqrt{\frac{rr_0}{r+r_0}n\lambda}$$

Let's estimate what would be radii of such zones:

$$\begin{cases} r_0 = r = 1\,m \\ n = 1 \\ \lambda = 550\,nm \end{cases} \Rightarrow R_1 = 0.5\,mm$$

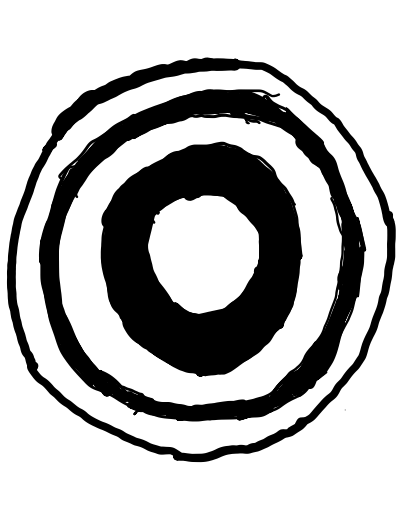
Now, let's find area of first zone:

$$G_n = 2\pi r_0 h = \frac{\pi r_0 r}{r_0+r} n\lambda$$



$$dG = G_n - G_{n-1} = \frac{\pi r_0 r}{r_0+r}\lambda$$

Area of Fresnel zone do not depend on n!



Thinner zones

Let's return to our integral. We were not able to solve it, because of K(R)

$$E_p = 2\pi\frac{r_0}{r_0+r}\frac{E_0'}{r_0}e^{ikr_0}\int_r^{R_{max}} K(R)e^{ikR}dR$$

What have changed now as we introduced Fresnel zones? Since the zone is small K(R) is constant within the zone.

$$K_n\int_{r+(n-1)\frac{\lambda}{2}}^{r+n\frac{\lambda}{2}} e^{ikR}dR = \frac{K_n}{ik}\left[e^{ikr}\left(e^{ikn\frac{\lambda}{2}} - e^{ikn\frac{\lambda}{2}}e^{-ik\frac{\lambda}{2}}\right)\right] =$$

$$= \frac{2K_n}{ik}(-1)^n e^{ikr}$$

↗ this is contribution of single Fresnel zone.

Now we need to sum up all zones.

$$E_p = \frac{4\pi i}{k}\cdot\frac{E_0'}{r_0+r}e^{ik(r_0+r)}(K_1 - K_2 + K_3 - K_4 + \ldots) =$$

$$= E_1 - E_2 + E_3 - E_4 + \ldots$$

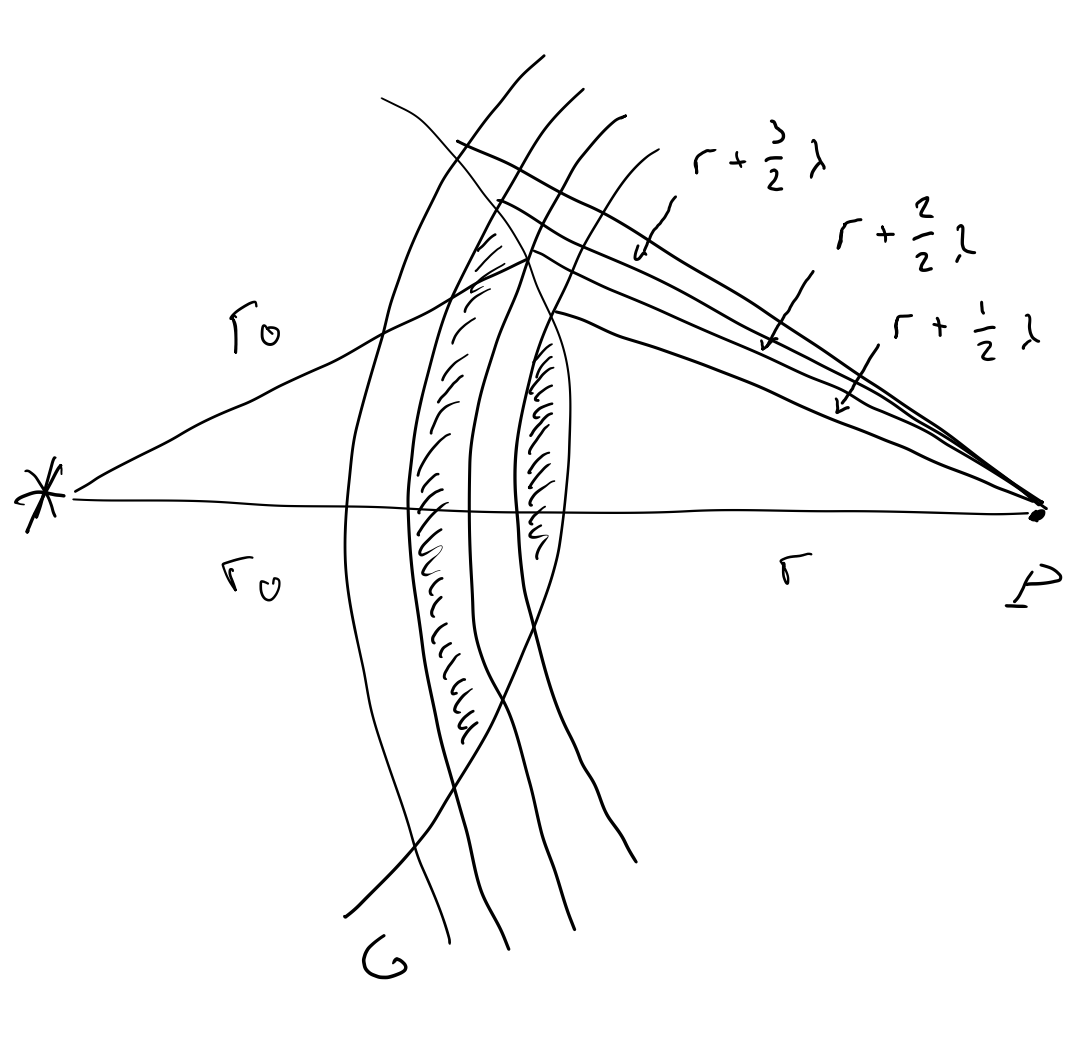
Remember $-1 = e^{i\pi}$. That means E_2 is in inverse phase to E_1, E_4 is in inverse phase to E_3, etc. Indeed, it should be like this as distance changes by $\frac{\lambda}{2}$.

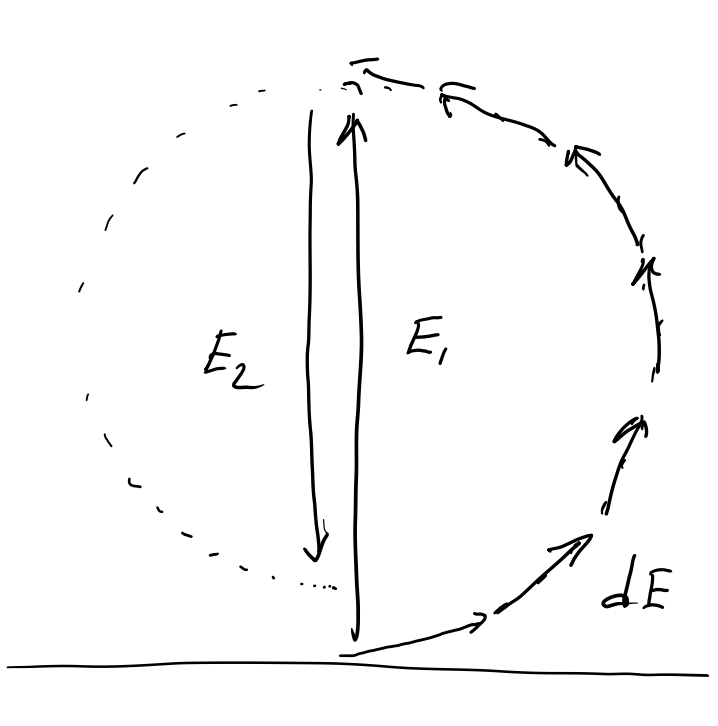
We can re-group here:

$$E_p = E_1 - (E_2 - E_3) - (E_4 - E_5) - \ldots < E_1$$

This is very interesting fact. What does it mean?

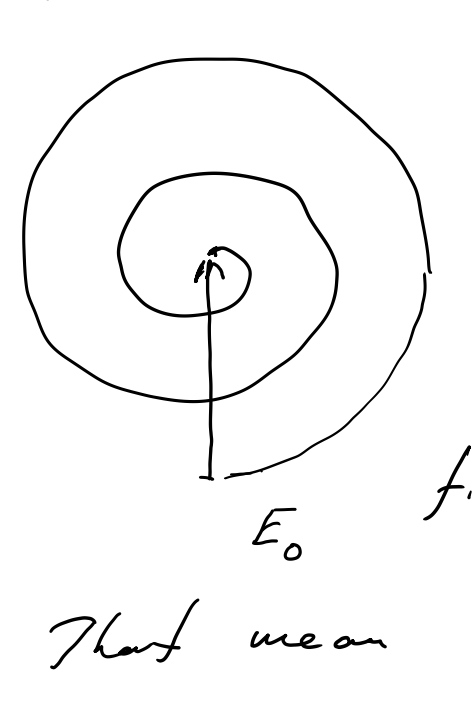
Let's use vector diagram.





Please, notice that the curve is not circle, but spiral. First, each vector dE has slightly smaller amplitude with distance increase. Moreover, phase difference is also not a constant.

If imagine there is no hole:



Important to note, $E_1 = 2E_0$

That means if only first zone is open, then $E = E_1$. That mean it will be $I_1 = 4I_0$.

The reason behind it is that all secondary sources are coherent.